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A variational approach to consistent single-phase upscaling

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SUMMARY

A consistent upscaling methodology is outlined, which bounds the stationary value of the original governing functional and highlights the source of errors that arise when an upscaled permeability field is substituted. The stationary value is identified as an important physical measure and the existence of bounds on this quantity enables confidence to be placed in upscaled predictions, including well productions, as opposed to empirical error estimates associated with conventional upscaling methods. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: upscaling; variational methods; comparison problems

1. INTRODUCTION

Upscaling is a technique currently employed by the oil industry in which the original geological data set is replaced by a lower resolution approximation enabling less-costly simulations to be run. The geological data is a piecewise constant set of permeability values that can extend over kilometres but vary over millimetres. Conventional upscaling methods replace this fine scale data by a coarsely defined piecewise constant function. Standard numerical methods can then be constructed on the areas of constant upscaled permeability, since an economic discretization can be made which coincides with the coarse permeability structure. The simulations are used to estimate well productions which are modelled as flux integrals over portions of the domain boundary. These integral quantities are of more interest to the oil industry than the general flow regime within the reservoir, as economic decisions will be influenced by them.

Conventional upscaling methods fall into two categories. The first category consists of explicit averaging methods including arithmetic, geometric and harmonic means, applied directly to regions of the original permeability field [1]. The second category consists of methods involving local flow simulations from which an effective coarse permeability is attributed to the region [2]. The upscaling methods in this category attempt to model the response of a flow over the original permeability data but require assumptions to be made regarding the boundary conditions for the local flow simulations. The boundary conditions assumed for these local

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flow simulations are normally a pressure potential with no-flow, or a linear pressure gradient, along the sides. The second category of upscaling methods can lead to diagonal permeability tensors by solving along each co-ordinate direction. A psuedo-local flow method bridging the two categories is also common. The arithmetic–harmonic method is implemented by taking the harmonic average along permeability strips in the direction of the flow and then arithmetically averaging the strips across the flow.

The reservoir is modelled using the steady-state diffusion equation for the pressure p, representing a single-phase incompressible flow obeying Darcy's law through media of permeability $\lambda(\mathbf{x})$. Thus the pressure and flux, $p(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$, respectively, satisfy

$$\mathbf{q}(\mathbf{x}) = -\lambda(\mathbf{x})\nabla p(\mathbf{x}) \quad \text{in } D \tag{1}$$

$$\nabla \cdot \mathbf{q}(\mathbf{x}) = 0 \quad \text{in } D \tag{2}$$

$$p(\mathbf{x}) = f(\mathbf{x}) \quad \text{on } C_p \tag{3}$$

$$\mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } C_q \tag{4}$$

where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal vector on C_q, and $C_p \cup C_q$ is the boundary of the given domain D. In the upscaled problem, $\lambda(\mathbf{x})$ is replaced by an upscaled permeability $\Lambda(\mathbf{x})$ with associated pressure and flux functions $\tilde{p}(\mathbf{x})$ and $\tilde{\mathbf{q}}(\mathbf{x})$, which are approximations to $p(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ and satisfy

$$\tilde{\mathbf{q}}(\mathbf{x}) = -\Lambda(\mathbf{x})\nabla \tilde{p}(\mathbf{x}) \quad \text{in } D \tag{5}$$

$$\nabla \cdot \tilde{\mathbf{q}}(\mathbf{x}) = 0 \quad \text{in } D \tag{6}$$

$$\tilde{p}(\mathbf{x}) = f(\mathbf{x}) \quad \text{on } C_p \tag{7}$$

$$\tilde{\mathbf{q}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } C_q \tag{8}$$

over the same domain. Conventional upscaling suffers from a discrepancy between the solutions p and $\tilde{p}$ of the two problems and this phenomenon compounds the error analysis of upscaled solutions.

In general, it is preferable that the upscaled permeability field is regarded as a function of the original permeability data only, as this allows the upscaling stage to be uniquely calculated in advance. Conventional upscaling methods possess this feature. Upscaling methods which result in a non-linear diffusion equation in which Λ is a function of both λ and $\tilde{p}$ do not permit the Λ calculation and pressure solution stages to be decoupled and therefore the size of the problem is not reduced. In addition, the non-linearities induced by $\Lambda(\lambda, \tilde{p})$ are likely to require an iterative procedure and the possibility arises that the upscaled system of equations becomes more expensive to solve than the original linear system.

In this paper we restrict ourselves to upscaling methods that can be separated from the pressure solution. In contrast to the existing methods, we adopt a more abstract approach and treat upscaling as a comparison problem. We introduce a functional which is stationary at the required set of governing equations. The stationary value of the functional is found to be a weighted flux integral over the boundary and this is used as the basis of an upscaling methodology. The new upscaling method aims to minimize the difference between the stationary value of the functional evaluated over the original and upscaled permeability data. Using the functional allows comparisons between different permeability fields to be made and a consistent upscaling method determined. The resulting consistent upscaling method constructs two permeability fields from which numerical bounds on the original stationary value can be calculated. The motivation and effectiveness of this new upscaling method will be discussed.

2. THE STATIONARY DIFFUSION FUNCTIONAL

The functional that yields the diffusion equation and boundary conditions when stationary, as well as the complementary variational principles given below, can be found in References [3, 4]. The diffusion equation and boundary conditions (1)–(4) are the natural conditions of the variational principle $\delta\mathscr{G}(\lambda, p, \mathbf{q}) = 0$ for the functional $\mathscr{G}(\lambda, p, \mathbf{q})$, where

$$\mathscr{G}(\lambda, p, \mathbf{q}) = \iint_D \left\{ \frac{\lambda^{-1}}{2}\mathbf{q}^2 + \mathbf{q}\cdot\nabla p \right\} \mathrm{d}\Omega - \int_{C_p} (p - f)\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\Sigma - \int_{C_q} pg\,\mathrm{d}\Sigma \tag{9}$$

and λ is a given symmetric positive-definite tensor. For the first variation of $\mathscr{G}(\lambda, p, \mathbf{q})$ is

$$\delta\mathscr{G}(\lambda, p, \mathbf{q}) = \iint_D \{\delta p(-\nabla\cdot\mathbf{q}) + \delta\mathbf{q}\cdot(\lambda^{-1}\mathbf{q} + \nabla p)\}\,\mathrm{d}\Omega \tag{10}$$

$$- \int_{C_p} (p - f)\delta\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\Sigma + \int_{C_q} \delta p(\mathbf{q}\cdot\mathbf{n} - g)\,\mathrm{d}\Sigma \tag{11}$$

and therefore the functional is stationary for any variations in p and $\mathbf{q}$ if and only if the diffusion equation and boundary conditions are satisfied. Thus, the problem of determining the solution of (1)–(4) is equivalent to finding the functions p and $\mathbf{q}$ which make $\mathscr{G}(\lambda, p, \mathbf{q})$ stationary.

The stationary value of the functional is found by substituting the stationary conditions into the functional, to give

$$\mathscr{G}(\lambda, p, \mathbf{q})_{\mathrm{Stat}} = \iint_D \left\{ \frac{\lambda^{-1}}{2}\mathbf{q}^2 + \mathbf{q}\cdot\nabla p \right\} \mathrm{d}\Omega - \int_{C_p} (p - f)\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\Sigma - \int_{C_q} pg\,\mathrm{d}\Sigma \tag{12}$$

$$= \iint_D \frac{1}{2}\mathbf{q}\cdot\nabla p\,\mathrm{d}\Omega - \int_{C_q} pg\,\mathrm{d}\Sigma \tag{13}$$

$$= \frac{1}{2}\int_{C_p} f\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\Sigma - \frac{1}{2}\int_{C_q} pg\,\mathrm{d}\Sigma \tag{14}$$

The stationary value is therefore a weighted integral of the flux over the boundary which is an important physical quantity representing well production in a reservoir model. The required weights in the integral can be chosen by considering the difference between two functionals [5]. The stationary value additionally benefits from second-order accuracy relative to first-order errors in p and $\mathbf{q}$, as the first-order variations of the functional vanish at the stationary point. The physical significance and advantageous accuracy of the stationary value of the functional make it an ideal basis for an upscaling method.

3. THE p AND $\mathbf{q}$ PRINCIPLES

The variational principle given above is a 'free principle' in the sense that the variations are unconstrained. The 'p principle' is formed by constraining the free principle $\mathcal{G}(\lambda, p, \mathbf{q})$ by the subset of the natural conditions

$$\mathbf{q} = -\lambda \nabla p \quad \text{in } D \tag{15}$$

$$p = f \quad \text{on } C_p \tag{16}$$

Substituting constraints (15) and (16) into $\mathcal{G}(\lambda, p, \mathbf{q})$, we obtain the '$p$ functional'

$$\mathcal{G}_p(\lambda, p) = -\iint_D \frac{1}{2}\lambda \nabla p \cdot \nabla p \, \mathrm{d}\Omega - \int_{C_q} pg \, \mathrm{d}\Sigma \tag{17}$$

which has (2) and (4) as natural conditions. The p principle retains the same stationary value as the free principle as the natural conditions of $\mathcal{G}_p(\lambda, p)$ along with the constraints imposed, (15) and (16), are equivalent to the natural conditions of the free principle. A lower bound on $\mathcal{G}(\lambda, p, \mathbf{q})_{\text{Stat}}$ is found by comparing the stationary values of $\mathcal{G}_p(\lambda, p)$ and $\mathcal{G}_p(\lambda, p_h)$, where p is the analytic solution satisfying the full set of stationary conditions, and p_h is *any* function satisfying constraint (16), since

$$\begin{aligned}
\mathcal{G}_p(\lambda, p) - \mathcal{G}_p(\lambda, p_h) &= \iint_D \left\{ \frac{1}{2}\lambda \nabla p_h \cdot \nabla p_h - \frac{1}{2}\lambda \nabla p \cdot \nabla p \right\} \mathrm{d}\Omega + \int_{C_q} g(p_h - p) \, \mathrm{d}\Sigma \\
&= \iint_D \left\{ \frac{1}{2}\lambda \nabla p_h \cdot \nabla p_h - \frac{1}{2}\lambda \nabla p \cdot \nabla p - \lambda \nabla p \cdot \nabla (p_h - p) \right\} \mathrm{d}\Omega \\
&= \iint_D \frac{\lambda}{2} (\nabla p_h - \nabla p)^2 \, \mathrm{d}\Omega \\
&\geqslant 0
\end{aligned} \tag{18}$$

Thus for any function p_h satisfying constraints (15) and (16), the value of the functional $\mathcal{G}_p(\lambda, p_h)$ cannot exceed the exact stationary value $\mathcal{G}(\lambda, p, \mathbf{q})_{\text{Stat}}$.

Similarly, the '$\mathbf{q}$ principle' is defined by constraining $\mathcal{G}(\lambda, p, \mathbf{q})$ by the subset of natural conditions

$$\nabla \cdot \mathbf{q} = 0 \quad \text{in } D \tag{19}$$

$$\mathbf{q} \cdot \mathbf{n} = g \quad \text{on } C_q \tag{20}$$

Substituting constraints (19) and (20) into $\mathscr{G}(\lambda, p, \mathbf{q})$ we obtain the '$\mathbf{q}$ functional'

$$\mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}) = \iint_D \frac{\lambda^{-1}}{2} \mathbf{q}^2 \, d\Omega + \int_{C_p} f \mathbf{q} \cdot \mathbf{n} \, d\Sigma \tag{21}$$

which has (1) and (3) as natural conditions. An upper bound on $\mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}}$ is found by comparing the stationary value of $\mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q})$ and $\mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}_h)$, where $\mathbf{q}$ is the analytic solution satisfying the full set of stationary conditions, and $\mathbf{q}_h$ is *any* function satisfying (19) and (20), since

$$\begin{aligned} \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}) - \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}_h) &= \iint_D \left\{ \frac{\lambda^{-1}}{2} \mathbf{q}^2 - \frac{\lambda^{-1}}{2} \mathbf{q}_h^2 \right\} d\Omega + \int_{C_p} f(\mathbf{q} - \mathbf{q}_h) \cdot \mathbf{n} \, d\Sigma \\ &= \iint_D \left\{ \frac{\lambda^{-1}}{2} \mathbf{q}^2 - \frac{\lambda^{-1}}{2} \mathbf{q}_h^2 + (\mathbf{q} - \mathbf{q}_h) \cdot \nabla p \right\} d\Omega \\ &= - \iint_D \frac{\lambda^{-1}}{2} (\mathbf{q} - \mathbf{q}_h)^2 \, d\Omega \\ &\leqslant 0 \end{aligned} \tag{22}$$

which can be considered as the dual of inequality (18). Since $\mathscr{G}_p(\lambda, p) = \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q})$ is the exact stationary value of the problem, we have

$$\max_{p_h} \mathscr{G}_p(\lambda, p_h) = \mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} = \min_{\mathbf{q}_h} \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}_h) \tag{23}$$

In particular, $\mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}}$ can be bounded above and below by approximate solutions p_h and $\mathbf{q}_h$ constructed in finite-dimensional spaces satisfying the appropriate constraints, and as the functionals $\mathscr{G}_p(\lambda, p_h)$ and $\mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}_h)$ are quadratic, the extrema can be found by weakly solving the appropriate stationary equations. The equalities in (23) then become

$$\mathscr{G}_p(\lambda, p_h)_{\text{Stat}} \leqslant \mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} \leqslant \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}_h)_{\text{Stat}} \tag{24}$$

where henceforth the subscript h denotes a finite-dimensional approximation.

The description of the rock permeability by a piecewise constant function is a feature of many current reservoir simulations. The discontinuities in the permeability function enable the simulation to be regarded as a multiple-domain problem with appropriate interfacial conditions. Crucially, the p and $\mathbf{q}$ principles can be applied across permeability discontinuities provided that the pressure is continuous in the p principle and the normal flux is continuous in the $\mathbf{q}$ principle. By enforcing the interfacial constraint on the pressure, the p principle satisfies the normal flux continuity at the interface as a natural condition. Similarly, enforcing flux continuity at the interface in the $\mathbf{q}$ principle ensures that the pressure continuity condition is satisfied as a natural condition. Note that in any approximate solution, the natural conditions will be satisfied in the weak sense and will therefore be a source of error.

4. CONSISTENT UPSCALING METHODS

The extension of the p and $\mathbf{q}$ principles to discontinuous λ fields allows coarse approximations to be constructed over the original permeability data. In part, this removes the motivation to upscale as a valid approximation of a practical size can be constructed. However, the disadvantage of coarse approximations defined over the original fine permeability data is that the interfacial conditions hold particularly weakly due to the typically low number of degrees of freedom in the solution relative to the number of permeability discontinuities. On the other hand, a coarse approximation constructed over a coarse permeability field can be made to coincide with the permeability discontinuities and the errors associated with satisfying the interfacial conditions will be reduced. A motivation to upscale is therefore to construct a coarse permeability field in such a way that the coarse discretization does indeed coincide with the permeability discontinuities and the errors incurred at the interfaces are minimized.

We therefore define the *aim* of upscaling to be the replacement of the original permeability field λ with a coarser description Λ such that

$$\mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} = \mathscr{G}(\Lambda, \tilde{p}, \tilde{\mathbf{q}})_{\text{Stat}} \tag{25}$$

where $\tilde{p}$ and $\tilde{\mathbf{q}}$ are the analytic solutions corresponding to the upscaled permeability field Λ for the original boundary conditions. Recognizing that we are unlikely to achieve the equality in (25), we introduce two coarsely defined permeability fields Λ^p and $\Lambda^{\mathbf{q}}$ which are constructed to satisfy the extremum principles and hence retain upper and lower bounds on $\mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}}$. Therefore, the consistent upscaling method can be considered as a pair of constrained optimization problems,

$$\min_{\Lambda^p}[\mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} - \mathscr{G}_p(\Lambda^p, \tilde{p}_h)_{\text{Stat}}] \ \text{ s.t. } \ \mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} - \mathscr{G}_p(\Lambda^p, \tilde{p}_h)_{\text{Stat}} \geqslant 0 \quad \forall \tilde{p}_h \tag{26}$$

$$\min_{\Lambda^{\mathbf{q}}}[\mathscr{G}_{\mathbf{q}}(\Lambda^{\mathbf{q}}, \tilde{\mathbf{q}}_h)_{\text{Stat}} - \mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}}] \ \text{ s.t. } \ \mathscr{G}_{\mathbf{q}}(\Lambda^{\mathbf{q}}, \tilde{\mathbf{q}}_h)_{\text{Stat}} - \mathscr{G}(\lambda, p, \mathbf{q})_{\text{Stat}} \geqslant 0 \quad \forall \tilde{\mathbf{q}}_h \tag{27}$$

The p and $\mathbf{q}$ principles present possible forms for Λ^p and $\Lambda^{\mathbf{q}}$. Considering the p principle, we can write

$$\begin{aligned}
\mathscr{G}_p(\lambda, p) - \mathscr{G}_p(\Lambda^p, \tilde{p}_h) &= \iint_D \left\{ \frac{\Lambda^p}{2} \nabla\tilde{p}_h \cdot \nabla\tilde{p}_h - \frac{\lambda}{2} \nabla p \cdot \nabla p \right\} \mathrm{d}\Omega + \int_{C_q} g(\tilde{p}_h - p)\, \mathrm{d}\Sigma \\
&= \iint_D \left\{ \frac{\Lambda^p}{2} \nabla\tilde{p}_h \cdot \nabla\tilde{p}_h - \frac{\lambda}{2} \nabla p \cdot \nabla p - \lambda\nabla p \cdot \nabla(\tilde{p}_h - p) \right\} \mathrm{d}\Omega \\
&= \iint_D \left\{ \frac{\lambda}{2} (\nabla\tilde{p}_h - \nabla p)^2 + \frac{1}{2} (\Lambda^p - \lambda)\nabla\tilde{p}_h \cdot \nabla\tilde{p}_h \right\} \mathrm{d}\Omega
\end{aligned} \tag{28}$$

Minimization and positivity of the right-hand side of (28) can be ensured by any of the following methods.

4.1. Inf–sup upscaling

Define Λ^p to be everywhere greater than or equal to λ, on each coarse region. The method can be implemented with various basis functions for Λ^p such that, whilst retaining $\Lambda^p \geqslant \lambda$ everywhere, the difference is also minimized.

4.2. Spectral upscaling

In the spectral upscaling method, we construct a consistent upscaled permeability field for all $\tilde{p}_h$ belonging to the space spanned by the chosen basis functions. To implement the method, we expand $\tilde{p}_h$ in terms of an appropriate set of basis functions $\{\phi_i\}$ as

$$\tilde{p}_h = \sum_{i=1}^{N} a_i \phi_i \tag{29}$$

and expand Λ^p in terms of a set of piecewise constant basis functions $\{\Theta_i\}$. We then determine γ such that positivity of the second term in (28) is satisfied in the sense that the matrix $\gamma R - S$ is positive semi-definite,

$$\mathbf{a}^{\mathrm{T}}(R - S)\mathbf{a} \geqslant 0 \quad \forall \mathbf{a} \tag{30}$$

where $\mathbf{a} = (a_1, \ldots, a_N)^{\mathrm{T}}$,

$$R_{ij} = \iint_{\Theta_i} \left\{ \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \alpha \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial x} \right) + \beta \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right\} \mathrm{d}\Omega \tag{31}$$

and

$$S_{ij} = \iint_{\Theta_i} \lambda \nabla \phi_i \cdot \nabla \phi_j \, \mathrm{d}\Omega \tag{32}$$

in 2D. This is accomplished by solving the generalized eigenvalue problem

$$S_r \mathbf{x} = \mu R_r \mathbf{x} \tag{33}$$

where S_r and R_r are S and R reduced by one row and column to remove the zero eigenvalue present. Choosing $\gamma = \max(\mu)$, the eigenvalue shift required for positivity is achieved since

$$(\gamma R_r - S_r)\mathbf{x} = (\gamma - \mu) R_r \mathbf{x} \tag{34}$$

The coefficients α and β are determined by clustering the eigenvalues v of $R_r^{-1} S_r$ in order that the spread of the eigenvalues $(\gamma - \mu)$ is minimized. The clustering is achieved through a simple numerical minimization of $\max(v) - \min(v)$. The upscaled permeability coefficient matrix Λ_i^p is then of the form

$$\Lambda_i^p = \begin{bmatrix} \gamma & \gamma\alpha \\ \gamma\alpha & \gamma\beta \end{bmatrix} \tag{35}$$

where α, β and γ are determined above. If only a scalar upscaling method is required, then the eigenvalue clustering stage can be avoided by setting $\alpha = 0$ and $\beta = 1$. Positivity of the second term in (28) follows directly from summing the positive contributions over $\{\Theta_i\}$.

4.3. Tuned basis functions

In addition to ensuring that the second term of (28) is positive, the first term can be minimized by selecting a set of basis functions that efficiently represent the pressure solution. Due to the flux continuity condition across the permeability discontinuities, the exact pressure gradient also experiences discontinuities at these interfaces. Therefore, in order to minimize the integral

$$\iint_D \frac{\lambda}{2}(\nabla\tilde{p}_h - \nabla p)^2\,\mathrm{d}\Omega \tag{36}$$

in (28), $\tilde{p}_h$ should also exhibit similar pressure gradient discontinuities. A set of tuned basis functions which satisfy the flux continuity condition to a greater extent can be constructed from local solutions over the λ field. This is implemented by expanding the tuned basis functions in terms of a set of fine basis functions $\{\sigma_j\}$ through

$$\hat{\phi}_i = \sum_j c_{ij}\sigma_j \tag{37}$$

and then solving

$$\iint_D \sigma_k \nabla\cdot\left(\lambda\nabla\sum_j a_{ij}\sigma_j - \nabla\phi\right)\mathrm{d}\Omega = 0 \quad \forall k \tag{38}$$

or

$$\iint_D \sigma_k \nabla\cdot\left(\lambda\nabla\sum_j a_{ij}\sigma_j - \lambda_{\mathrm{eff}}\nabla\phi\right)\mathrm{d}\Omega = 0 \quad \forall k \tag{39}$$

for the unknown coefficients of c_{ij}, where

$$\hat{\phi}_i(\mathbf{x}) = \phi_i(\mathbf{x}) \quad \mathbf{x}\in C_p \cup C_q \tag{40}$$

and if the basis functions ϕ_i have compact support

$$\hat{\phi}_i(\mathbf{x}) = \phi_i(\mathbf{x}) \quad \text{if } \phi_i(\mathbf{x}) = 0 \tag{41}$$

The use of any conventional upscaling method to produce a symmetric tensor Λ_{eff} allows the anisotropy of the media to be further incorporated into the tuned basis functions. Having solved for the coefficients a_{ij} the basis functions require normalizing and, as they are not orthogonal, they must also be adjusted so that they sum to unity. As a result of this adjustment, they no longer satisfy (38) or (39) exactly, but still retain much of the required details.

Consistent $\mathbf{q}$ principle upscaling methods may be constructed from the dual of (28) written as

$$\mathscr{G}_{\mathbf{q}}(\Lambda^{\mathbf{q}}, \tilde{\mathbf{q}}_h) - \mathscr{G}_{\mathbf{q}}(\lambda, \mathbf{q}) = \iint_D \left\{\frac{\lambda^{-1}}{2}(\mathbf{q} - \tilde{\mathbf{q}}_h)^2 + \frac{1}{2}((\Lambda^{\mathbf{q}})^{-1} - \lambda^{-1})\tilde{\mathbf{q}}_h^2\right\}\mathrm{d}\Omega \tag{42}$$

from which direct parallels with the methods above can be constructed in order to ensure positivity and minimize the right-hand side.

5. DISCUSSION OF CONSISTENT METHODS

In conventional upscaling, the discrepancy between the solutions p and $\tilde{p}$ of the two problems is difficult to estimate. For example, an error measure on the flux consists of a standard discretization error due to the difference between the numerical solution $\Lambda\nabla\tilde{p}_h$ and the exact solution of the upscaled problem $\Lambda\nabla\tilde{p}$, and a consistency error $\Lambda\nabla\tilde{p} - \lambda\nabla p$, due to the difference between the exact upscaled and original solutions. The flux error therefore satisfies the triangle inequality

$$\|\Lambda\nabla\tilde{p}_h - \lambda\nabla p\| \leqslant \|\Lambda\nabla\tilde{p}_h - \Lambda\nabla\tilde{p}\| + \|\Lambda\nabla\tilde{p} - \lambda\nabla p\| \tag{43}$$

In contrast, the consistent upscaling method enables the error in the upscaled solution to be both quantified and bounded. The convergence of the upper and lower bounds on the stationary value, calculated using various upscaling methods is plotted in Figure 1. The consistency errors incurred by the different methods are illustrated by the different values that the bounds converge to. The solutions were constructed over the random permeability field shown with the boundary conditions $p=1$ on $x=0$, $p=0$ on $x=1$ and $\mathbf{q}\cdot\mathbf{n}=0$ on $y=0,1$. The stationary value is equal to half the outward normal flux integral over the inlet $x=0$. The upscaled permeabilities were defined over a 2×2 grid and the numerical solutions calculated on a $N\times N$ grid using bilinear rectangular elements for the p principle and a stream function discretized using bilinear rectangular elements for the $\mathbf{q}$ principle. From Figure 1, it can be

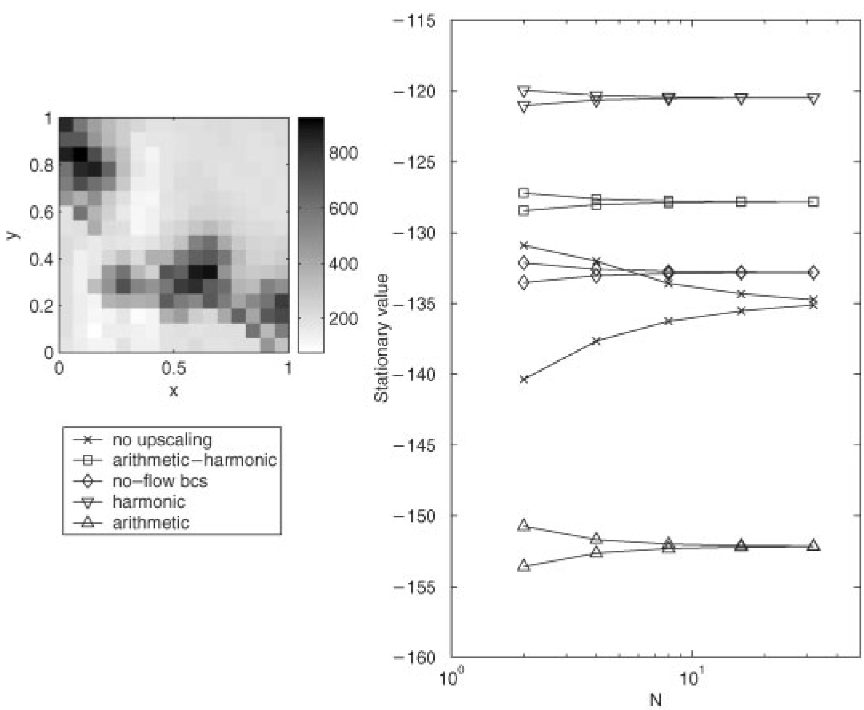

Figure 1. Convergence of bounds and comparison of consistent methods.

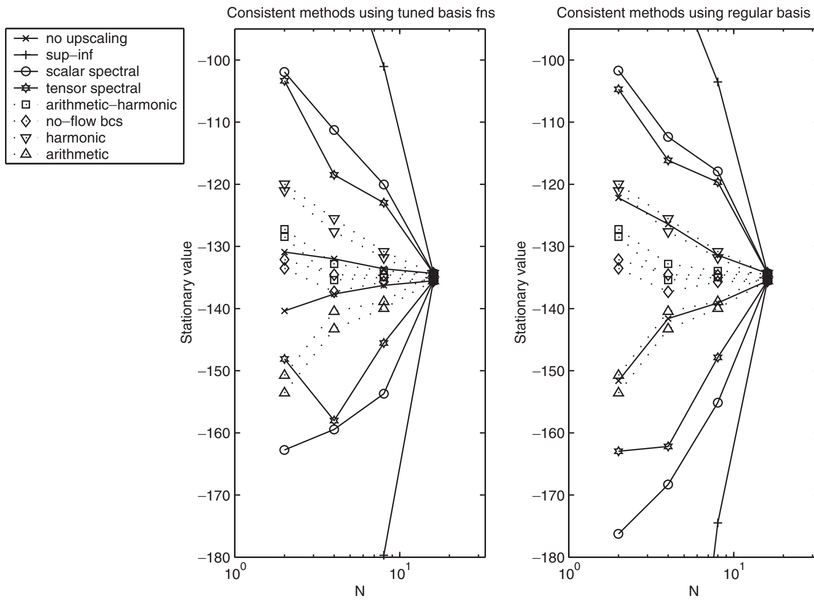

Figure 2. Comparison of upscaling methods.

seen that the standard upscaling methods based on local flow solutions or assumptions produce the better approximations to the stationary value, and that the tight bounds obtained with the upscaled permeability fields illustrate the advantage in having similarly resolved numerical discretizations and permeability fields.

The comparison method employed in the consistent upscaling method illuminates the non-linear nature of upscaling in which Λ^p is an arithmetic mean weighted by the solution $(\nabla\tilde{p})^2$ and $\Lambda^{\mathbf{q}}$ is as a harmonic mean weighted by $\tilde{\mathbf{q}}^2$. As a consequence of constructing upscaling methods that are independent of the solutions $\tilde{p}$ and $\tilde{\mathbf{q}}$, in general, an error will always be generated by the second term in (28) and the corresponding $\mathbf{q}$ principle (42). As a result, the consistent upscaling method can only ever be as accurate as solving the original problem using the same basis functions. One exception in which upscaling is reduced to a linear problem is when $\nabla\tilde{p}$ and $\tilde{\mathbf{q}}$ are modelled as piecewise constant functions in which case Λ^p simplifies to the arithmetic mean of λ defined over regions of constant $\nabla\tilde{p}$, and $\Lambda^{\mathbf{q}}$ is the harmonic mean defined similarly.

The performance of the consistent upscaling methods is shown in Figure 2. The solutions were constructed with the same boundary conditions and permeability field as used for Figure 1, with the upscaled permeabilities and numerical solutions calculated on a $N \times N$ grid. The inf–sup upscaling was implemented using a piecewise constant expansion and from Figure 2 it can be seen that the method gives very weak bounds unless the variations in λ are small over the area being upscaled. The accuracy of the method would be improved if a higher-order expansion were used that could fit the λ data more closely. The spectral methods are more competitive with the extra degrees of freedom in the symmetric tensor case helping to tighten the bounds. The restriction of $\tilde{p}$ and $\tilde{\mathbf{q}}$ to a known space enables the spectral method to approach the non-linear cases $\Lambda^{p}(\lambda, \tilde{p})$ and $\Lambda^{\mathbf{q}}(\lambda, \tilde{\mathbf{q}})$ whilst still enabling the upscaling stage to be calculated in advance. In contrast to the inf–sup method, the spectral method permits $\Lambda^{p} < \max(\lambda)$ within each upscaling region, which helps to tighten the bounds. Again higher-order expansions for Λ^{p} and $\Lambda^{\mathbf{q}}$ could be considered. The use of the tuned basis functions with the consistent upscaling methods also helps to tighten the bounds and solving over the original permeability field with the tuned basis functions is particularly effective. The conventional upscaling methods were implemented using regular basis functions and again the methods based on local flow calculations or assumptions appeared to be the more accurate. The accuracy of these local flow methods may in part be due to the solution p behaving roughly linearly, and therefore the boundary conditions assumed in the local flow calculations may be relatively accurate. With a more complex set of boundary conditions or geometry, the local flow simulations may not be so representative of what is happening in the domain and the accuracy of these methods may suffer. In contrast the consistent upscaling methods always retain bounds on the stationary value and the tightness of the bounds is a measure of the error in the $\tilde{p}$ and $\tilde{\mathbf{q}}$ solutions, and an indication as to how well the flow has been resolved.

The consistent upscaling method also has a natural application to λ fields with bounded uncertainties, as Λ^{p} and $\Lambda^{\mathbf{q}}$ can be defined from the minimum and maximum functions of λ. The effect of the degree and location of the uncertainty could then be investigated.

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